NUMERICAL CALCULATION OF THE TUNE SPREAD INDUCED BY TRANSVERSE SPACE CHARGE IN A SYNCHROTRON

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1. Introduction.

Transverse space charge forces are included in 6 dimensional tracking codes like *Accsim* [1] or *Track2D* [2]. Their inclusion is essential for the correct simulation of high intensity machines like the accumulator ring for the proposed National Spallation Neutron Source (NSNS).

In dealing with space charge, the immediate problem is that the on line calculation can be very time consuming and therefore impractical when a very large number of representative particles are used. Accordingly, one should devise some method to reach a compromise between the accuracy of the results and computation speed.

Two distinct, but related problems should be addressed.

(i) Particles are subject to the electromagnetic forces generated by the magnets in the ring and by the RF system, and to the forces due to the space charge. Forces by magnets provide the confinement of the beam and forces by the RF produce synchrotron oscillations. Space charge forces are responsible for diffusion that may lead to beam losses.

(ii) Betatron oscillations are affected by space charge forces. Particles in different positions in the beam have different betatron frequencies. Some may encounter resonances and may be rapidly lost. It is important to evaluate the evolution of tune distribution within the beam and to calculate the growth rate of possible destructive resonances.

In the following we will briefly revise the theory of transverse space charge treatment in circular accelerators, and discuss a computation strategy and the approximations involved. We will address the problem of tune spread calculation in the beam for different charge distributions. Examples for the NSNS will be shown. Here, we limit ourselves to the case of a very large vacuum chamber, i.e. when walls are far away.

2. The elementary theory of transverse space charge.

A good reference is Alex Chao's book [3]. The simplest expression for the space charge induced tune shift, for a beam of constant circular cross section around the accelerator and uniform density, when all walls are removed, is the so-called Bruck formula

$$\Delta\nu = -\frac{r_0 N}{\beta^2\gamma^3\nu}\frac{R}{2\pi a^2}, \tag{1}$$

where N is the number of particles in the beam, R is the average machine radius, β

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and γ the relativistic factors, ν the betatron tune and a the beam cross radius. r_0 is the classical particle radius, expressed by (in MKS units)

$$r_0 = \frac{\mu_0}{4\pi}\frac{e^2}{m_0}. \tag{2}$$

For protons, it is $r_0 = 1.534698\ 10^{-18} m$.

A "back of an envelope" estimate for the NSNS (1 GeV), with a circumference $2\pi R = 209m,\ a = 2cm,\ \beta = 0.8750256,\ \gamma = 2.065789,$

for a betatron frequency of $\nu = 3.75$ and a number of protons per turn $N = 2\ 10^{14}$, Eq.(1), gives

$\Delta\nu = -0.1024$

A more elaborate expression than Eq.(1), including also image charges and currents on pipe walls can be found, according to Laslett [3] for non uniform charge distribution in the cross section (e.g. Gaussian).

Far from walls, the space charge forces can be calculated by an integration on the actual charge distribution. If the local beam charge density is $\rho(Q)$, the electric field component along $\hat{r}$, due to a charge volume $dQ = dx_s dy_s dz_s$, where x_s and y_s are the transverse coordinates and z_s the longitudinal coordinate, is

$$E_r = \frac{1}{\epsilon_0}\int \frac{\rho}{r^3}\hat{r}\, dQ = \frac{1}{\epsilon_0} g(P), \tag{3}$$

with r the distance of the field point $P \equiv (x, y, z)$ from the source charge volume at $Q \equiv (x_s y_s z_s)$

$$r^2 = (x - x_s)^2 + (y - y_s)^2 + \frac{(z - z_s)^2}{\gamma^2}. \tag{4}$$

The factor $1/\gamma^2$ in Eq.(4) is there because in the longitudinal direction the field appears flattened due to relativistic contraction.

The magnetic field $\vec{B}$ on P due to a thin charge filament passing through Q is perpendicular to $\vec{E}$. The total force, electric plus magnetic along $\hat{r}$, on a charge e at P, recalling that $\epsilon_0\mu_0 = c^{-2}$ and $\mu_0 = 4\pi\ 10^{-7}$ in MKS, is given by

$$F_r = \frac{\mu_0 e^2 c^2 n}{\gamma^2}\int \frac{\rho(Q)}{r^3}\hat{r}\, dQ, \tag{5}$$

with ne the charge contained in some beam volume V.

The normalization condition for the charge density is

$$\frac{1}{V}\int \rho(Q)\, dQ = Ne. \tag{6}$$

The space charge force of Eq.(5) will produce a tune shift $\Delta\nu$ for the particle at P both in the x and y directions. Different particles in the beam will have different tune shifts, so we will consider a tune spread $\delta\nu$ in the beam. To calculate the tune spread, consider in the smooth approximation the betatron oscillation equation of a given particle in the transverse coordinate r, either x or y

$$\frac{d^2 r}{dz^2} + K_r r = f(r, z), \tag{7}$$

with

$$f(r,z) = \frac{F_r}{m_0\gamma\beta^2c^2}. \tag{8}$$

The space charge force is zero in the center of mass of the beam and is approximately anti symmetric, because the force is expected to be repulsive outwards on both sides. Then, near the center the function f will be in general linear with r. Let us write it there as

$$f(r \approx 0, z) = \delta K\, r. \tag{9}$$

Since in the smooth approximation it is

$$K \approx (\frac{\nu}{R})^2, \tag{10}$$

obtain for $r \approx 0$

$$\Delta\nu = -\frac{R^2}{2\nu}\frac{f(r,z)}{r}, \tag{11}$$

which is the tune shift for a particle near the center of mass of the beam, and often the maximum tune shift.

Introducing in Eq.(11) the integral g defined in Eq.(3), and recalling that the number of particle in a longitudinal chunk of the beam of length δz is

$$ne = Ne\frac{\delta\, z}{2\pi R},$$

the following expression for the maximum tune shift is obtained, to be compared with Eq.(1)

$$\Delta\nu = -\frac{r_0 N}{\beta^2\gamma^3\nu}R\frac{g\,\delta z}{r}. \tag{12}$$

Eq.(7) is in general a non linear equation that generates oscillations with a variety of frequencies. There are many ways to find the frequencies. The most direct is to integrate numerically the equation for many different particles. Approximate analytical solutions can be found either by expanding f in powers of the transverse coordinate and interpreting the results as multipole errors driven. We may also expand f in harmonics and trying to apply the theory of coupled Hill equations (equations in x and in y are copled through the function f) to find Floquet exponents for instability growth rates etc.

In this note, we want only discuss a numerical approach to the problem, that can be implemented as a subroutine of a simulation code.

3. Numerical calculation of space charge forces and tune spread.

The numerical work was done with three related simple computer codes

qtrack, qbin, qshift,

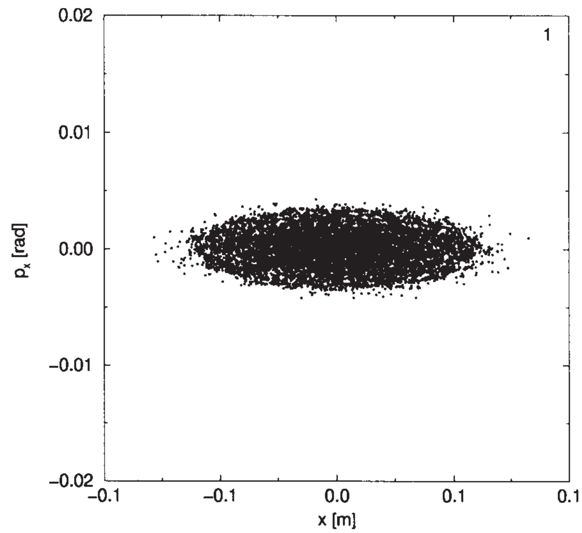

Fig. 1. Population #1 in x, p_x.

each driven by an input file *file_name.in* The code reads input files (created by [the] other codes), and produces output files, for graphic plotting.

3.1. Code qtrack

This code reads the following input

A *population* file containing a gaussian distribution of transverse phase space coordinates for N_{macro} macroparticles x, p_x, y, p_y. In the following examples we used 10^4 particles.

A *lattice* file containing the ring lattice, in this case the NSNS.

Then, the code builds the 4×4 optical transfer matrices and start tracking particles over many turns. The number of particles is increased at each turn to simulate multi turn injection. The center of the injection phase space $x_0, p_{x,0} . y_0, p_{y,0}$ is being moved during injection to achieve different filling pattern of the machine acceptance.

We have considered four cases:

1- x_0 and y_0 move from $6cm$ to $0cm$ during the injection. This produces a gaussian like distribution filling the acceptance.

2- x_0 and y_0 move from $6cm$ to $4cm$. This produces a "smoke ring", a phase space distribution in the acceptance with a crater in the middle.

3- x_0 and y_0 move from $6cm$ to $2cm$. This produces a thinner smoke ring.

4- x_0 and y_0 don't move from $6cm$.. This produces the thinnest smoke ring.

The distribution of particles in the (x, p_x) phase space at the end of the injection process is shown in Figs.1,2,3,4, respectively. In the (y, p_y) phase space the pattern is substantially identical

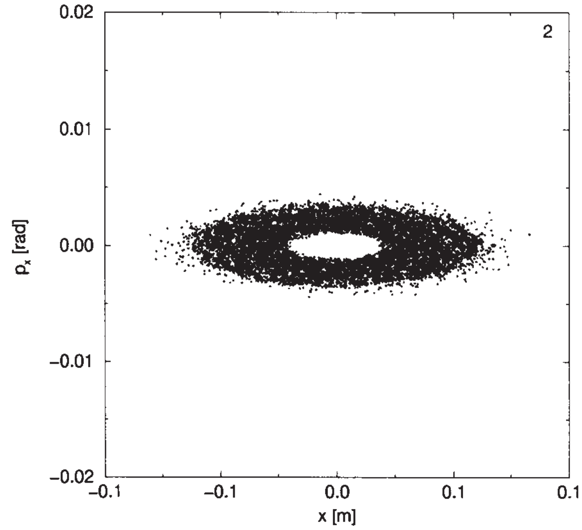

Fig. 2. Ring #2 in x, p_x.

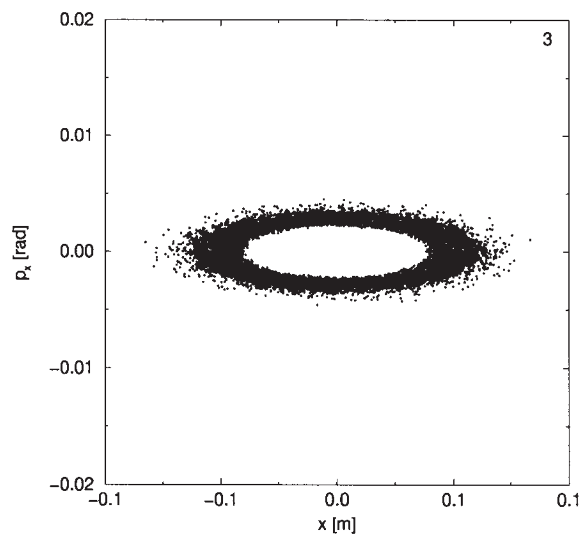

Fig. 3. Ring #3 in x, p_x.

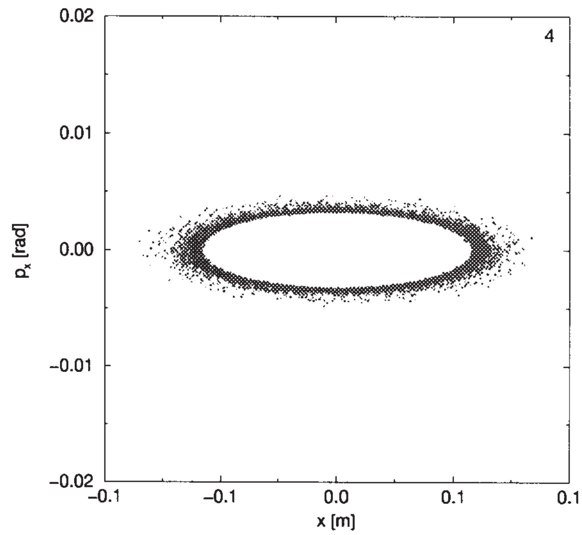

Fig. 4. Ring #4 in x, p_x.

3.2. Code qbin

This code reads as input the *population* created by *qtrack* after a certain number of turn. Then, by binning, the charge distribution $\rho_\perp(x_s, y_s)$ is built, to be used as a source for the electromagnetic space charge force calculation on each particle of the beam located at P. The particle density corresponding to the four cases of Figs.1 through 4 is shown in Figs.5,6,7,8, respectively. In the following calculations for the forces at P we assume that the beam cross section remains constant over a length δz, and then write the charge density as

$$\rho(Q) = \rho_{||}\rho_\perp(x_s, y_s).$$

The code produces the force field components in the transverse directions $F_x(x, y)$ and $F_y(x, y)$ and writes them to an output file.

From Eq.(5), the x field component can be derived as follows, by first performing the integral with respect to the longitudinal coordinate z_s (the calculation for the y componernt is similar). Start from

$$g_x(P) = \int dQ \frac{\rho(Q)(x - x_s)}{r^3} = \rho_{||} \iint dx_s\, dy_s\, \rho_\perp(x, y)(x - x_s) \int dz_s \frac{1}{r^3}, \qquad (13)$$

where

$$dQ = dx_s dy_s dz_s$$

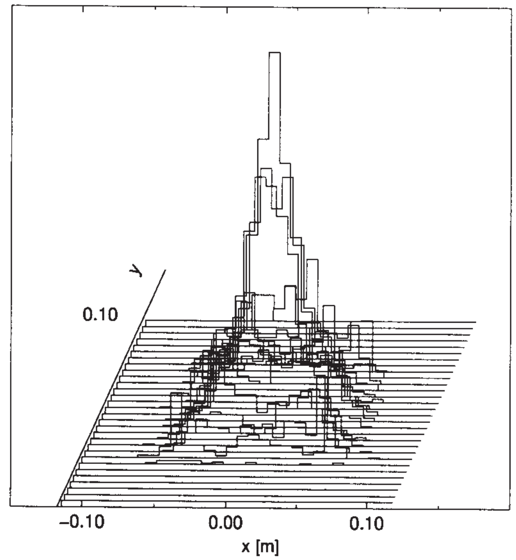

Fig. 5. Population #1. Charge density.

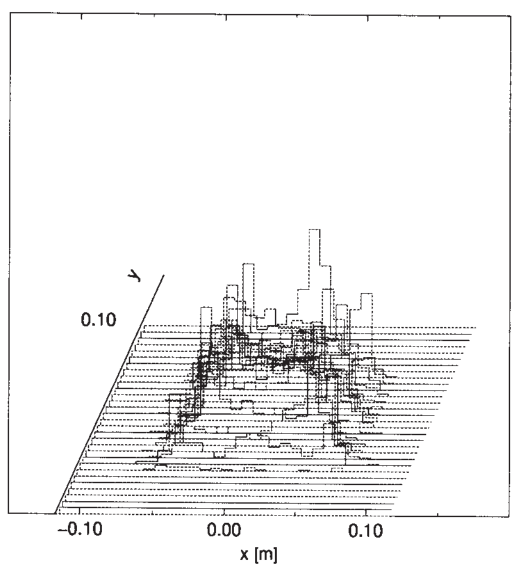

Fig. 6. Ring #2. Charge density.

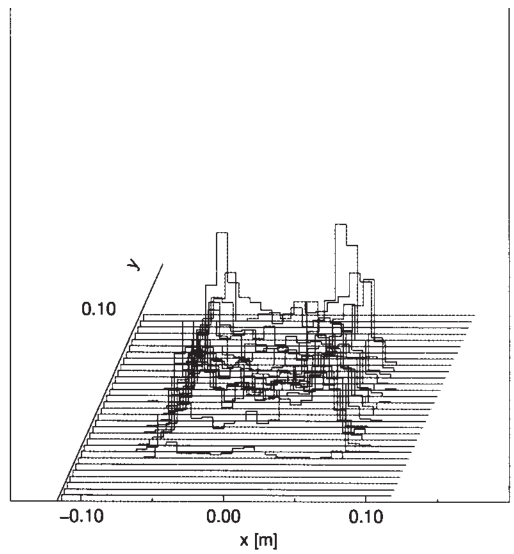

Fig. 7. Ring #3. Charge density.

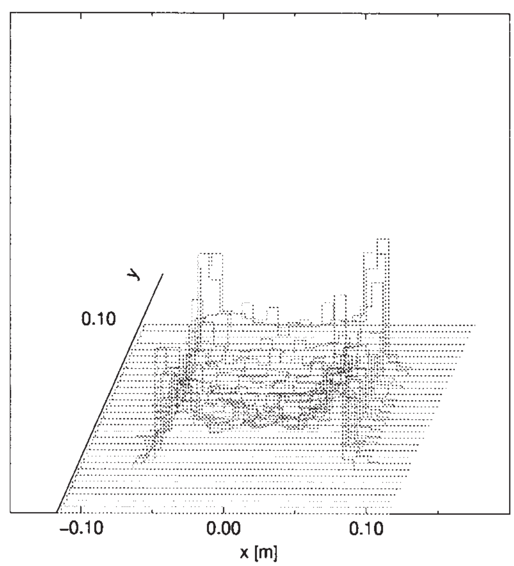

Fig. 8. Ring #4. Charge density.

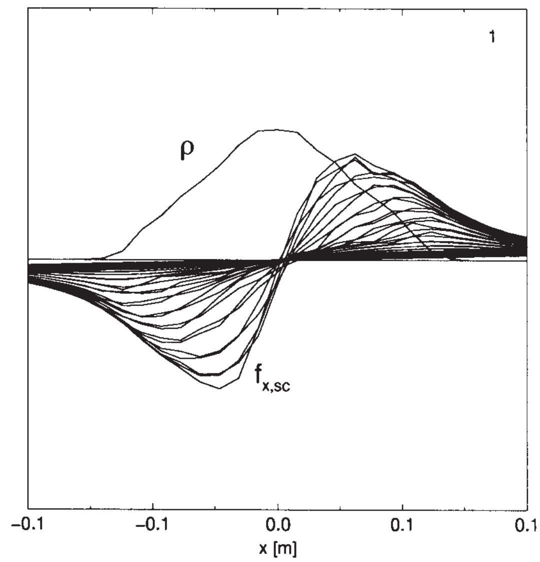

Fig. 9. Population #1. Density and Space charge force vs. x for various y.

and

$$r^2 = r_\perp^2 + \frac{(z - z_s)^2}{\gamma^2}, r_\perp^2 = (x - x_s)^2 + (y - y_s)^2.$$

The inner integral is immediate. Now, since in the following arguments the longitudinal coordinate disappears, we can assume it to be zero, i.e. $z = 0$, and finally write

$$g_x(P) = \rho_{||}\delta z \iint dx_s\, dy_s \frac{\rho_\perp(x,y)}{r_\perp^2\sqrt{r_\perp^2 + \delta z^2/(2\gamma)^2}}(x - x_s)^2. \tag{14}$$

The space charge force field in the x direction and the beam density profile corresponding to Figs.1 through 4 are shown in Figs.9,10,11,12, respectively. In the figures each force curve corresponds to the force behaviour for one of 31 slices across the beam along a constant y. The figures clearly show that the force derivative in the center of the beam decreases moving from a gaussian type particle distribution (case 1 of *qtrack*) to a smoke ring distribution (cases 2,3,4). So, the tune shift for particles near the beam center of mass is expected to decrease. What happens of particles near the edge of the beam has to be calculated.

3.3. Code qshift

This code reads the final *population* created by *qtrack* and the space charge field data, *scfield*, created by *qbin*, then integrates the differential equations of betatron motion of Eq.(7) for a large number of turns and for a large number of representative particles. During integration, values of x and y are computed at constant intervals

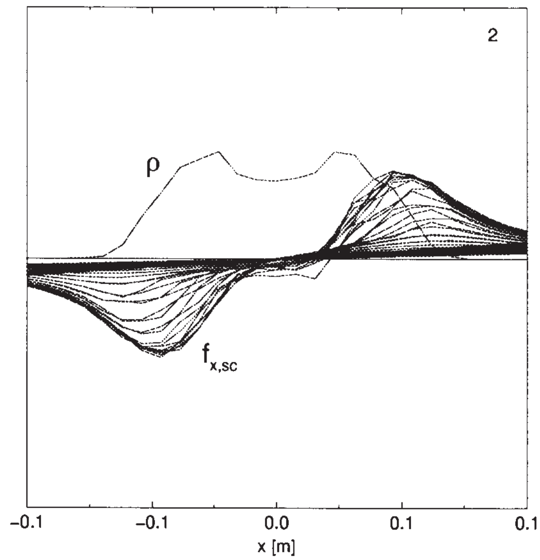

Fig. 10. Ring #2. Density and Space charge force vs. x for various y.

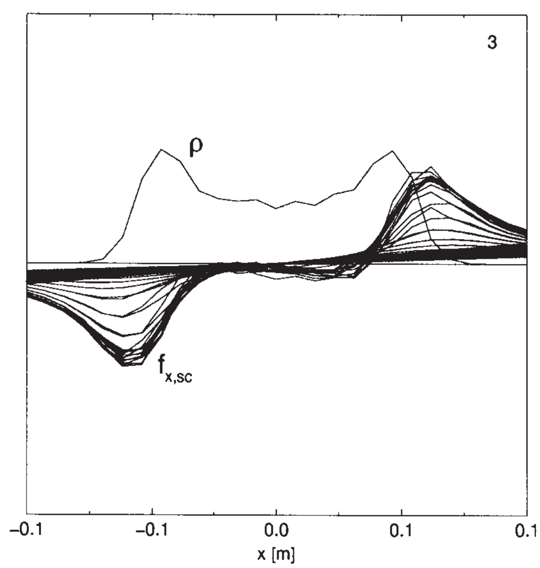

Fig. 11. Ring #3. Density and Space charge force vs. x for various y.

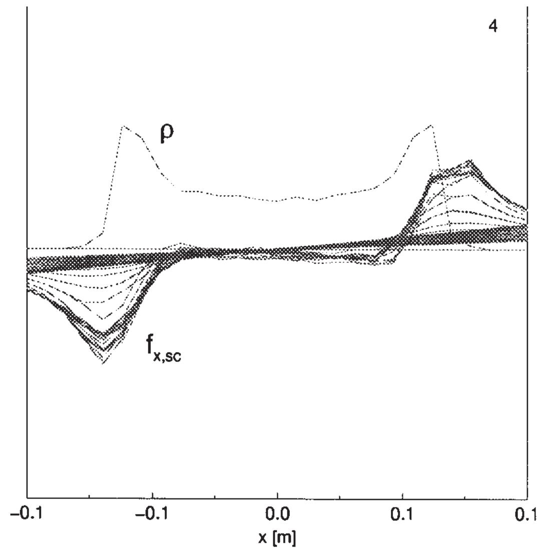

Fig. 12. Ring #4. Density and Space charge force vs. x for various y.

of z. Finally, a FFT is done on the orbits, and the power spectrum is written to an output file *tune*, convoluted over all the particles.

Typically, $n_F = 4096$ points are used for the FFT over ≈ 200 turns with a $\delta z = 10m$ interval between points (the NSNS ring circumference is 209 m). This in turn represents a relative betatron frequency resolution of

$$\frac{\delta\omega_\beta}{\omega_0} = \frac{2\pi R}{\delta z\, n_F} \approx 0.005,$$

where ω_0 is the revolution frequency in the ring.

Results for 256 particles are shown in Figs.13,14,15,16, for the four population cases computed by *qtrack*, respectively. The different spectral line are for increasing number of particles in the beam, and betatron tunes $\nu_x = 3.7, \nu_y = 3.8$.

An important result shown by the figures is that in a ring configuration the tune shift, averaged over many particles in the beam, is considerably less than in a filled emittance.

A second important effect is also apparent and expected. Because of the coupling between the x and y motion forced by the space charge term in the r.h.s. of Eq.(7), frequencies characteristic of the x oscillations appear in the y plane and viceversa. Fig.17 shows the tune shift on a group of 5 particles at various positions in the distribution of Fig.1. Beat frequencies are clearly seen. The figure shows also that the tune shift is larger for particles near the center, as it should.

The distribution of the highest peaks in the tune chart $\nu_x - \nu_y$ for the spectrum of Fig.13 is finally shown in Fig.18.

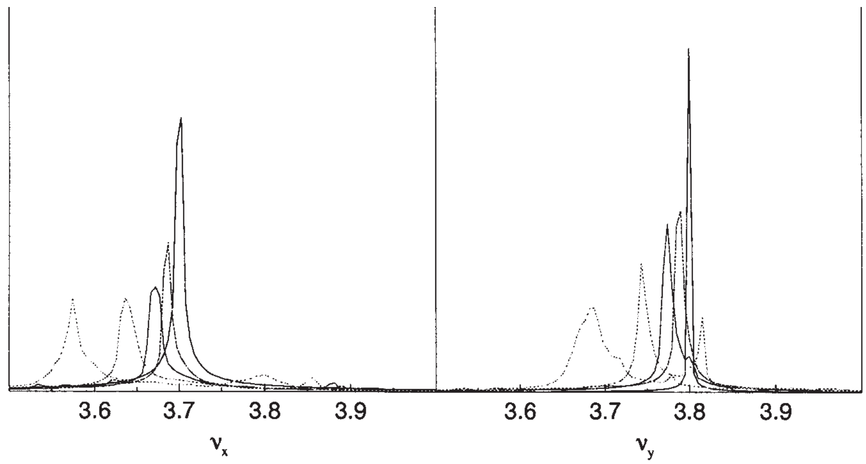

Fig. 13. Population #1, 256 particles. Tune shift for $N = 0, 0.25, 0.5, 1, 2\,10^{14}$.

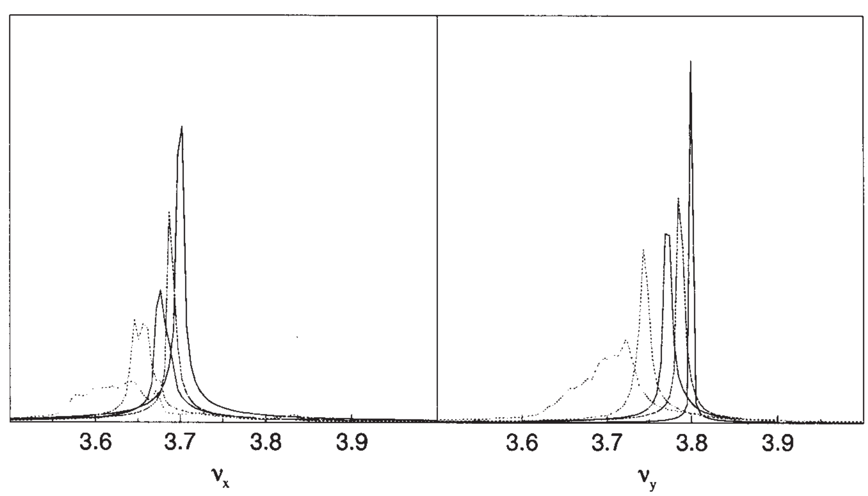

Fig. 14. Ring #2, 256 particles. Tune shift for $N = 0, 0.25, 0.5, 1, 2\,10^{14}$.

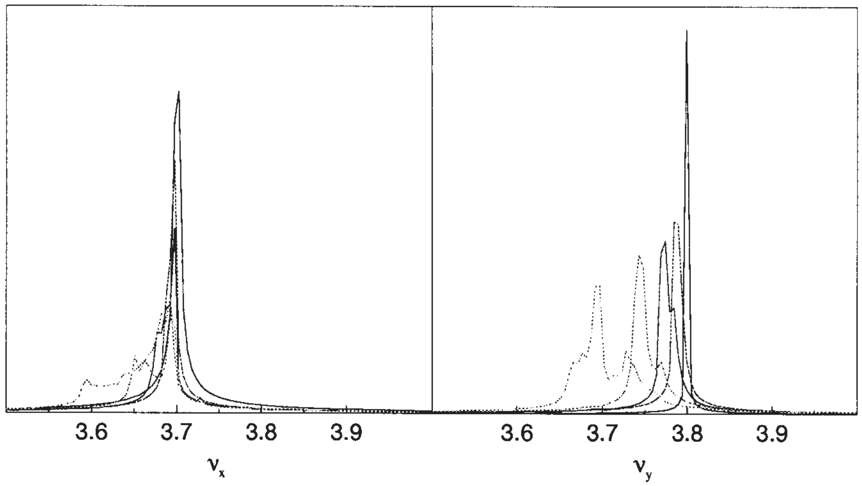

Fig. 15. Ring #3, 256 particles. Tune shift for $N = 0, 0.25, 0.5, 1, 2\,10^{14}$.

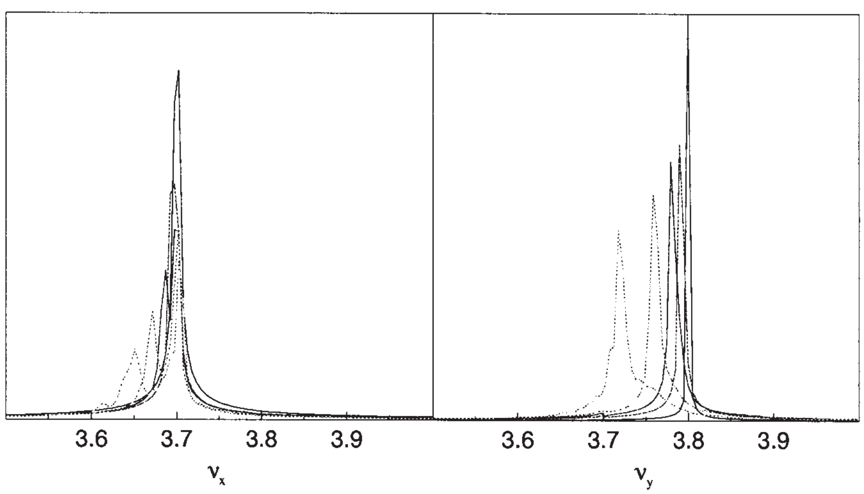

Fig. 16. Ring #4, 256 particles. Tune shift for $N = 0, 0.25, 0.5, 1, 2\,10^{14}$.

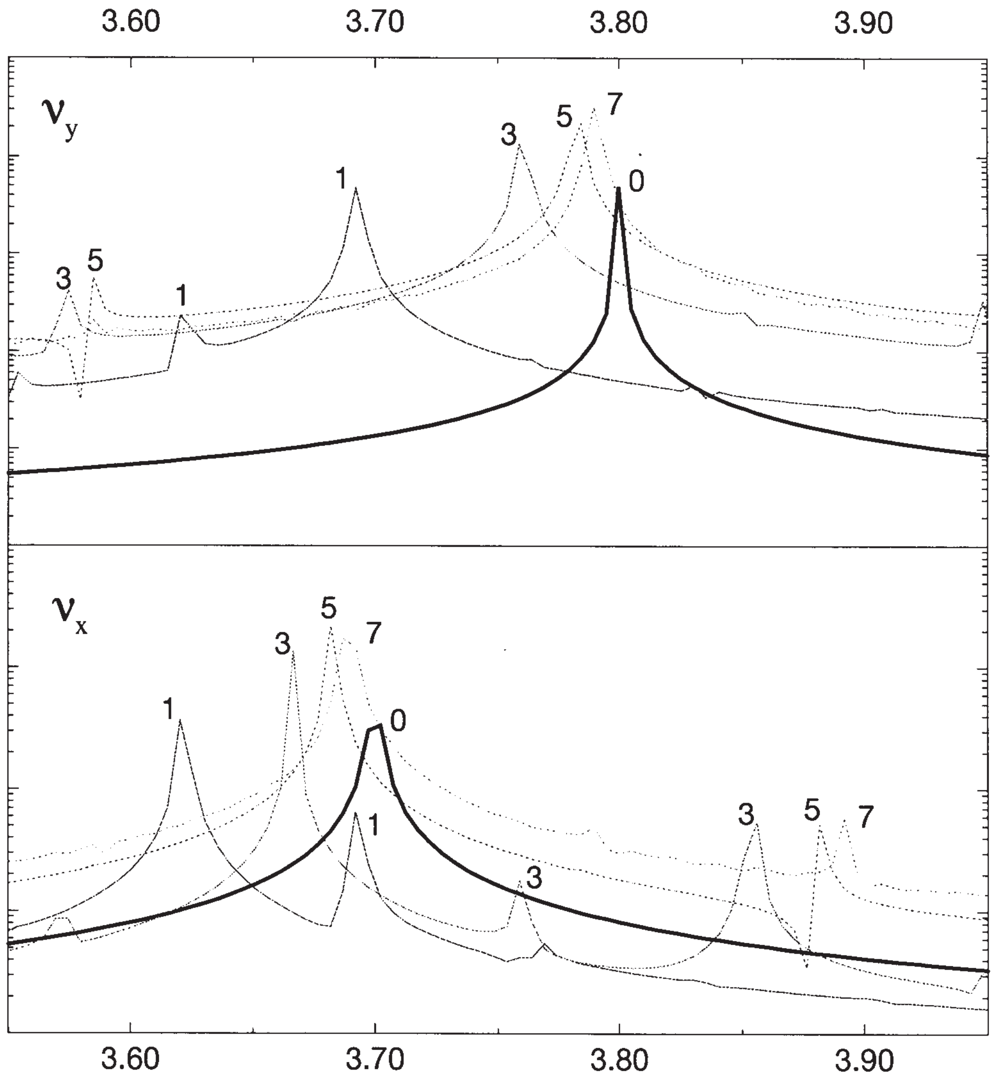

Fig. 17. Tune shifts for various amplitude oscillations.Coupling frequencies between x and y modes are apparent. The calculation is done for $N = 1\,10^{14}$ and different particles, with positions labeled in cm on the curves.

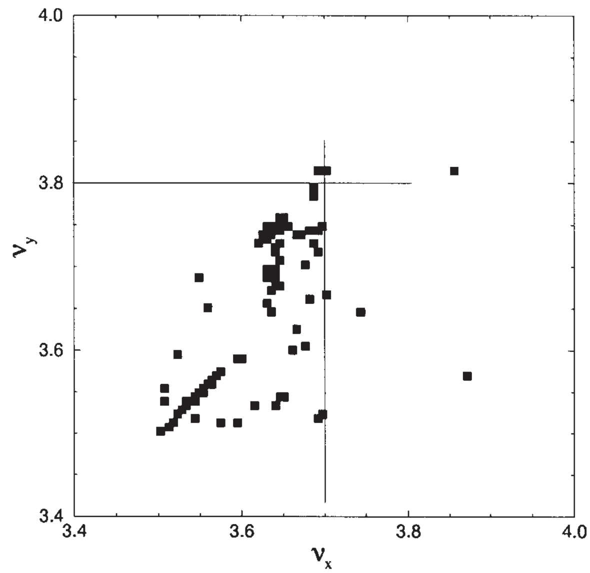

Fig. 18. Distribution of peaks in the tune chart. 256 particles. Population #1.

4. References.

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