Three-Dimensional Field Expansions in Magnets:A primer

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C. J. Gardner

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ALTERNATING GRADIENT SYNCHROTRON DEPARTMENT
BROOKHAVEN NATIONAL LABORATORY
UPTON, NEW YORK 11973

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1 Introduction

In small rings such as the SNS accumulator ring, where the ratio of length to aperture of the magnets is relatively small, the magnetic field near the ends of the magnets requires special attention. Unlike the field in the region well inside the magnets, the field near the ends depends on the position along the longitudinal axis of the magnet and has a component parallel to the axis. Here one can not simply expand the field in terms of two-dimensional normal and skew multipoles; a full three-dimensional expansion is required. In this note we review the standard expansions in rectangular and cylindrical coordinates carried out by Brown [1] and Danby [2] respectively. The expansion in rectangular coordinates may be more familiar to those who work with tracking codes such as TRANSPORT or MAD, while the expansion in cylindrical coordinates may be more familiar to those who design and measure magnets. Both expansions are useful and it is instructive to write them down in one place and to compare the expansion coefficients.

2 Expansion in Rectangular Coordinates

Inside the aperture of a magnet the curl of the static magnetic field, **B**, is zero which implies that **B** can be expressed as the gradient of a scalar potential, ϕ. Thus

$$\mathbf{B} = \nabla\phi, \tag{1}$$

and since the divergence of **B** must also vanish we have

$$\nabla^2 \phi = 0. \tag{2}$$

The desired expansion of the magnetic field follows from the expansion of ϕ about the longitudinal axis of the magnet. (The longitudinal axis coincides with the center of the magnet aperture and lies in the horizontal midplane of the magnet.) We shall employ the orthogonal set of unit vectors $(\mathbf{x}, \mathbf{y}, \mathbf{z})$, where $\mathbf{z}$ points along the longitudinal axis and $\mathbf{x}$ and $\mathbf{y}$ point respectively along the horizontal and vertical directions transverse to the axis. The orientation of the vectors is such that $\mathbf{x} \times \mathbf{y} = \mathbf{z}$. Coordinates x and y specify the distances from the longitudinal axis along $\mathbf{x}$ and $\mathbf{y}$ respectively. z specifies the distance along $\mathbf{z}$ measured from some reference point on the axis. In terms of these vectors and coordinates we have

$$\mathbf{B} = \nabla \phi = \mathbf{x}\frac{\partial \phi}{\partial x} + \mathbf{y}\frac{\partial \phi}{\partial y} + \mathbf{z}\frac{\partial \phi}{\partial z} \tag{3}$$

and

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \tag{4}$$

2.1 Solution of Laplace Equation

For each z along the axis of the magnet we can expand the potential ϕ as follows:

$$\phi(x, y, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn} \frac{x^n}{n!} \frac{y^m}{m!} \tag{5}$$

where the coefficients C_{mn} are functions of z. Putting this expansion into (4) we find the following recursion relation for the coefficients C_{mn}:

$$C_{m+2,n} = -C_{m,n+2} - C''_{mn} \tag{6}$$

where the primes denote differentiation with respect to z. Putting (5) into (3) we find that the components of **B** in the **x**, **y**, and **z** directions are respectively

$$\begin{aligned}
B_x(x, y, z) &= \frac{\partial \phi}{\partial x} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{m,n+1} \frac{x^n}{n!} \frac{y^m}{m!} \\
B_y(x, y, z) &= \frac{\partial \phi}{\partial y} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{m+1,n} \frac{x^n}{n!} \frac{y^m}{m!}
\end{aligned}$$

$$B_z(x,y,z) \quad = \quad \frac{\partial\phi}{\partial z} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C'_{m,n}\frac{x^n}{n!}\frac{y^m}{m!}. \tag{7}$$

This is the desired expansion of the magnetic field about the longitudinal axis.

2.2 Multipole Coefficients

Differentiating the expansions for B_y and B_x with respect to x and evaluating them at $x = y = 0$, we find

$$C_{1n} = \left(\frac{\partial^n B_y}{\partial x^n}\right)_{x=y=0}, \quad C_{10} = B_y(0,0,z) \tag{8}$$

and

$$C_{0,n+1} = \left(\frac{\partial^n B_x}{\partial x^n}\right)_{x=y=0}, \quad C_{01} = B_x(0,0,z). \tag{9}$$

These coefficients give the strengths of the normal and skew multipole fields respectively. The longitudinal component of the field on the magnet axis is given by the last of equations (7) with $x = y = 0$. Thus

$$B_z(0,0,z) = C'_{00} \tag{10}$$

and we see that the coefficient C'_{00} gives the strength of the solenoid field. It follows from the recursion relation (6) that all of the coefficients in the magnetic field expansion can be expressed in terms of the multipole and longitudinal field coefficients and their derivatives with respect to z.

To simplify notation, let us define

$$B_n = C_{1n}, \quad A_n = C_{0,n+1} \tag{11}$$

for the normal and skew multipole coefficients. On the horizontal midplane (i.e. for $y = 0$) we then have

$$\begin{aligned} B_x(x,0,z) &= \sum_{n=0}^{\infty} A_n \frac{x^n}{n!} \\ B_y(x,0,z) &= \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \\ B_z(x,0,z) &= C'_{00} + \sum_{n=0}^{\infty} A'_n \frac{x^{n+1}}{(n+1)!}. \end{aligned} \tag{12}$$

Here we see that if the skew coefficients (A_n) are all zero, then on the midplane we have $B_x = 0$ and $B_z = C'_{00}$. If, in addition, the solenoid coefficient C'_{00} is zero, then B_y is the only nonzero field component on the midplane (i.e. the field is perpendicular to the midplane). This is why the B_n are called "normal" coefficients. Similarly, if the B_n are all zero, then on the midplane we have $B_y = 0$ and B_x is nonzero. This is why the A_n are called "skew" coefficients.

2.3 Expansion of B to Fifth-order

Using (11) and the recursion relation (6) we find that the coefficients required to expand the field to fifth order are

$$C_{20} = -A_1 - C''_{00}, \quad C_{21} = -A_2 - A''_0, \quad C_{22} = -A_3 - A''_1 \tag{13}$$

$$C_{23} = -A_4 - A''_2, \quad C_{24} = -A_5 - A''_3 \tag{14}$$

$$C_{30} = -B_2 - B''_0, \quad C_{31} = -B_3 - B''_1, \tag{15}$$

$$C_{32} = -B_4 - B''_2, \quad C_{33} = -B_5 - B''_3 \tag{16}$$

$$C_{40} = -C_{22} - C''_{20} = A_3 + 2A''_1 + C''''_{00} \tag{17}$$

$$C_{41} = -C_{23} - C''_{21} = A_4 + 2A''_2 + A''''_0 \tag{18}$$

$$C_{42} = -C_{24} - C''_{22} = A_5 + 2A''_3 + A''''_1 \tag{19}$$

$$C_{50} = -C_{32} - C''_{30} = B_4 + 2B''_2 + B''''_0 \tag{20}$$

$$C_{51} = -C_{33} - C''_{31} = B_5 + 2B''_3 + B''''_1 \tag{21}$$

$$C_{60} = -C_{42} - C''_{40} = -A_5 - 3A''_3 - 3A''''_1 - C''''''_{00}. \tag{22}$$

Using these coefficients in (7) and collecting terms of equal order we then obtain the following expansions of B_x, B_y, and B_z in x and y. The superscripts N and S designate the parts of the expansions containing normal and skew coefficients respectively.

2.3.1 B_x expansion ($B_x = B_x^N + B_x^S$)

$$
\begin{aligned}
B_x^N &= B_1 y \\
&+ B_2 xy \\
&+ B_3(3x^2y - y^3)/6 - B_1'' y^3/6 \\
&+ B_4(x^3y - xy^3)/6 - B_2'' xy^3/6 \\
&+ B_5(5x^4y - 10x^2y^3 + y^5)/120 \\
&+ B_3''(y^5 - 5x^2y^3)/60 + B_1'''' y^5/120
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
B_x^S &= A_0 \\
&+ A_1 x \\
&+ A_2(x^2 - y^2)/2 - A_0'' y^2/2 \\
&+ A_3(x^3 - 3xy^2)/6 - A_1'' xy^2/2 \\
&+ A_4(x^4 - 6x^2y^2 + y^4)/24 \\
&+ A_2''(y^4 - 3x^2y^2)/12 + A_0'''' y^4/24 \\
&+ A_5(x^5 - 10x^3y^2 + 5xy^4)/120 \\
&+ A_3''(xy^4 - x^3y^2)/12 + A_1'''' xy^4/24
\end{aligned} \tag{24}
$$

2.3.2 B_y expansion ($B_y = B_y^N + B_y^S$)

$$
\begin{aligned}
B_y^N &= B_0 \\
&+ B_1 x \\
&+ B_2(x^2 - y^2)/2 - B_0'' y^2/2 \\
&+ B_3(x^3 - 3xy^2)/6 - B_1'' xy^2/2 \\
&+ B_4(x^4 - 6x^2y^2 + y^4)/24 \\
&+ B_2''(y^4 - 3x^2y^2)/12 + B_0'''' y^4/24 \\
&+ B_5(x^5 - 10x^3y^2 + 5xy^4)/120 \\
&+ B_3''(xy^4 - x^3y^2)/12 + B_1'''' xy^4/24
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
B_y^S &= -A_1 y - C_{00}'' y \\
&- A_2 xy - A_0'' xy \\
&+ A_3(y^3 - 3x^2y)/6
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad A_1''(2y^3 - 3x^2y)/6 + C_{00}''''y^3/6 \\
&+ \quad A_4(xy^3 - x^3y)/6 \\
&+ \quad A_2''(2xy^3 - x^3y)/6 + A_0''''xy^3/6 \\
&- \quad A_5(y^5 - 10x^2y^3 + 5x^4y)/120 \\
&- \quad A_3''(3y^5 - 20x^2y^3 + 5x^4y)/120 \\
&+ \quad A_1''''(10x^2y^3 - 3y^5)/120 - C_{00}''''''y^5/120
\end{aligned} \tag{26}
$$

2.3.3 B_z expansion ($B_z = B_z^N + B_z^S$)

$$
\begin{aligned}
B_z^N \quad &= \quad B_0'y \\
&+ \quad B_1'xy \\
&+ \quad B_2'(3x^2y - y^3)/6 - B_0'''y^3/6 \\
&+ \quad B_3'(x^3y - xy^3)/6 - B_1'''xy^3/6 \\
&+ \quad B_4'(5x^4y - 10x^2y^3 + y^5)/120 \\
&+ \quad B_2'''(y^5 - 5x^2y^3)/60 + B_0'''''y^5/120
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
B_z^S \quad &= \quad C_{00}' \\
&+ \quad A_0'x \\
&+ \quad A_1'(x^2 - y^2)/2 - C_{00}'''y^2/2 \\
&+ \quad A_2'(x^3 - 3xy^2)/6 - A_0'''xy^2/2 \\
&+ \quad A_3'(x^4 - 6x^2y^2 + y^4)/24 \\
&+ \quad A_1'''(y^4 - 3x^2y^2)/12 + C_{00}'''''y^4/24 \\
&+ \quad A_4'(x^5 - 10x^3y^2 + 5xy^4)/120 \\
&+ \quad A_2'''(xy^4 - x^3y^2)/12 + A_0'''''xy^4/24
\end{aligned} \tag{28}
$$

Note that when the fields are independent of z, all of the A and B terms in (23–28) involving derivatives with respect to z are zero and the field expansion contains only "pure" multipole terms. If, in addition, the C_{00}' term (responsible for solenoid fields) is zero, then B_z is zero and the fields are transverse to the axis of the magnet. This is the situation well inside a non-solenoid magnet, away from the ends.

3 Expansion in Cylindrical Coordinates

Let us now introduce cylindrical coordinates r and θ such that

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{29}$$

The expansion (5) then becomes

$$\phi(r,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}\frac{r^n}{n!}\frac{r^m}{m!}\cos^n\theta\sin^m\theta \tag{30}$$

which we may write in the form

$$\phi(r,\theta,z) = G_{00} + \sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\{F_{lm}\sin m\theta + G_{lm}\cos m\theta\}r^l. \tag{31}$$

The coefficients F_{lm} and G_{lm} are functions of z and are responsible respectively for the normal and skew multipole fields (as we shall show). The terms in (31) that are proportional to $\sin m\theta$ or $\cos m\theta$ are called $2m$-pole terms because they arise from magnets with $2m$ poles.

Defining unit vectors

$$\mathbf{r} = \mathbf{x}\cos\theta + \mathbf{y}\sin\theta, \quad \mathbf{\Theta} = -\mathbf{x}\sin\theta + \mathbf{y}\cos\theta \tag{32}$$

we have

$$\mathbf{B} = \nabla\phi = \mathbf{r}\frac{\partial\phi}{\partial r} + \mathbf{\Theta}\frac{1}{r}\frac{\partial\phi}{\partial\theta} + \mathbf{z}\frac{\partial\phi}{\partial z}, \tag{33}$$

and the components of $\mathbf{B}$ in the $\mathbf{r}$, $\mathbf{\Theta}$, and $\mathbf{z}$ directions are respectively

$$\begin{aligned}
B_r &= \frac{\partial\phi}{\partial r} = \sum_{l=1}^{\infty}\sum_{m=0}^{\infty} l\{F_{lm}\sin m\theta + G_{lm}\cos m\theta\}r^{l-1} \\
B_\theta &= \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \sum_{l=1}^{\infty}\sum_{m=0}^{\infty} m\{F_{lm}\cos m\theta - G_{lm}\sin m\theta\}r^{l-1} \\
B_z &= \frac{\partial\phi}{\partial z} = G'_{00} + \sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\{F'_{lm}\sin m\theta + G'_{lm}\cos m\theta\}r^l.
\end{aligned} \tag{34}$$

The Laplace equation becomes

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{35}$$

3.1 Solution of Laplace Equation

Putting the expansion (31) into (35) we obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = \sum_{l=1}^{\infty}\sum_{m=0}^{\infty} l^2\{F_{lm}\sin m\theta + G_{lm}\cos m\theta\}r^{l-2} \tag{36}$$

$$\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \sum_{l=1}^{\infty}\sum_{m=0}^{\infty} -m^2\{F_{lm}\sin m\theta + G_{lm}\cos m\theta\}r^{l-2} \tag{37}$$

$$\frac{\partial^2\phi}{\partial z^2} = G''_{00} + \sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\{F''_{lm}\sin m\theta + G''_{lm}\cos m\theta\}r^{l} \tag{38}$$

and, in order to satisfy (35) for all r, we must have

$$\sum_{m=0}^{\infty}(1-m^2)\{F_{1m}\sin m\theta + G_{1m}\cos m\theta\} = 0, \tag{39}$$

$$G''_{00} + \sum_{m=0}^{\infty}(4-m^2)\{F_{2m}\sin m\theta + G_{2m}\cos m\theta\} = 0, \tag{40}$$

and, for $l \geq 1$,

$$\begin{aligned} &\sum_{m=0}^{\infty}\{(l+2)-m^2\}\{F_{l+2,m}\sin m\theta + G_{l+2,m}\cos m\theta\} \\ + &\sum_{m=0}^{\infty}\{F''_{lm}\sin m\theta + G''_{lm}\cos m\theta\} = 0. \end{aligned} \tag{41}$$

Then, in order to satisfy (39–41) for all θ, we must have

$$G_{10} = 0, \quad G_{12} = G_{13} = G_{14} = G_{15} = \ldots = 0 \tag{42}$$

$$F_{12} = F_{13} = F_{14} = F_{15} = \ldots = 0, \tag{43}$$

$$4G_{20} + G''_{00} = 0, \tag{44}$$

$$G_{21} = 0, \quad G_{23} = G_{24} = G_{25} = G_{26} = \ldots = 0, \tag{45}$$

$$F_{21} = 0, \quad F_{23} = F_{24} = F_{25} = F_{26} = \ldots = 0, \tag{46}$$

and, for $l \geq 1$,

$$\{(l+2)^2 - m^2\}F_{l+2,m} = -F''_{lm} \tag{47}$$

and

$$\{(l+2)^2 - m^2\} G_{l+2,m} = -G''_{lm}. \tag{48}$$

These equations give the complete solution of the Laplace equation in cylindrical coordinates.

3.2 Expansion Coefficients

On the horizontal midplane ($y = 0$) we have $\theta = 0$ and equations (34) become

$$\begin{aligned}
B_r(r,0,z) &= \sum_{l=1}^{\infty}\sum_{m=0}^{\infty} l G_{lm} r^{l-1} \\
B_\theta(r,0,z) &= \sum_{l=1}^{\infty}\sum_{m=0}^{\infty} m F_{lm} r^{l-1} \\
B_z(r,0,z) &= G'_{00} + \sum_{l=1}^{\infty}\sum_{m=0}^{\infty} G'_{lm} r^{l}.
\end{aligned} \tag{49}$$

Noting that $B_r = B_x$ and $B_\theta = B_y$ on the midplane, and comparing equations (49) with (12), we see that the normal and skew coefficients, B_n and A_n, depend only on the coefficients F_{lm} and G_{lm} respectively. We therefore refer to the F and G coefficients as normal and skew coefficients respectively.

It follows from (42–46) and the recursion relations (47–48) that many of the F and G coefficients are zero. In fact, for $m > l$, we have $F_{lm} = 0$ and $G_{lm} = 0$. The nonzero coefficients to order 10 are listed in Tables I and II.

Table I: Nonzero F_{lm} Coefficients				
F_{11}				
F_{22}				
F_{31}	F_{33}			
F_{42}	F_{44}			
F_{51}	F_{53}	F_{55}		
F_{62}	F_{64}	F_{66}		
F_{71}	F_{73}	F_{75}	F_{77}	
F_{82}	F_{84}	F_{86}	F_{88}	
F_{91}	F_{93}	F_{95}	F_{97}	F_{99}
$F_{10,2}$	$F_{10,4}$	$F_{10,6}$	$F_{10,8}$	$F_{10,10}$

Table II: Nonzero G_{lm} Coefficients					
G_{11}					
G_{20}	G_{22}				
G_{31}	G_{33}				
G_{40}	G_{42}	G_{44}			
G_{51}	G_{53}	G_{55}			
G_{60}	G_{62}	G_{64}	G_{66}		
G_{71}	G_{73}	G_{75}	G_{77}		
G_{80}	G_{82}	G_{84}	G_{86}	G_{88}	
G_{91}	G_{93}	G_{95}	G_{97}	G_{99}	
$G_{10,0}$	$G_{10,2}$	$G_{10,4}$	$G_{10,6}$	$G_{10,8}$	$G_{10,10}$

Putting numbers for l and m into the recursion relations (47–48), we obtain

$$F_{31} = -\frac{1}{8}F''_{11}, \quad F_{51} = -\frac{1}{24}F''_{31}, \quad F_{71} = -\frac{1}{48}F''_{51} \tag{50}$$

$$F_{42} = -\frac{1}{12}F''_{22}, \quad F_{62} = -\frac{1}{32}F''_{42}, \quad F_{82} = -\frac{1}{60}F''_{62} \tag{51}$$

$$F_{53} = -\frac{1}{16}F''_{33}, \quad F_{73} = -\frac{1}{40}F''_{53}, \quad F_{93} = -\frac{1}{72}F''_{73} \tag{52}$$

$$F_{64} = -\frac{1}{20}F''_{44}, \quad F_{84} = -\frac{1}{48}F''_{64} \tag{53}$$

$$F_{75} = -\frac{1}{24}F''_{55}, \quad F_{95} = -\frac{1}{56}F''_{75} \tag{54}$$

$$F_{86} = -\frac{1}{28}F''_{66}. \tag{55}$$

Thus we see that the nonzero coefficients F_{lm} may all be expressed in terms of derivatives (with respect to z) of the coefficients F_{mm}. The corresponding equations for the G coefficients are the same except for the additional equations

$$G_{20} = -\frac{1}{4}G''_{00}, \quad G_{40} = -\frac{1}{16}G''_{20}, \quad G_{60} = -\frac{1}{36}G''_{40}, \tag{56}$$

$$G_{80} = -\frac{1}{64}G''_{60}, \quad G_{10,0} = -\frac{1}{100}G''_{80}. \tag{57}$$

3.3 Expansion of B to Fifth-order

Using the nonzero coefficients in (34) and collecting terms, we obtain the expansions of B_r, B_θ, and B_z to 5th order in r. The superscripts N and S designate the parts of the expansions containing F (Normal) and G (Skew) coefficients respectively.

3.3.1 B_r expansion ($B_r = B_r^N + B_r^S$)

$$\begin{aligned} B_r^N &= (F_{11} + 3F_{31}r^2 + 5F_{51}r^4)\sin\theta \\ &+ (2F_{22}r + 4F_{42}r^3 + 6F_{62}r^5)\sin 2\theta \\ &+ (3F_{33}r^2 + 5F_{53}r^4)\sin 3\theta \\ &+ (4F_{44}r^3 + 6F_{64}r^5)\sin 4\theta \\ &+ 5F_{55}r^4 \sin 5\theta \\ &+ 6F_{66}r^5 \sin 6\theta \end{aligned} \tag{58}$$

$$\begin{aligned} B_r^S &= 2G_{20}r + 4G_{40}r^3 + 6G_{60}r^5 \\ &+ (G_{11} + 3G_{31}r^2 + 5G_{51}r^4)\cos\theta \\ &+ (2G_{22}r + 4G_{42}r^3 + 6G_{62}r^5)\cos 2\theta \\ &+ (3G_{33}r^2 + 5G_{53}r^4)\cos 3\theta \\ &+ (4G_{44}r^3 + 6G_{64}r^5)\cos 4\theta \\ &+ 5G_{55}r^4 \cos 5\theta \\ &+ 6G_{66}r^5 \cos 6\theta \end{aligned} \tag{59}$$

3.3.2 B_θ expansion ($B_\theta = B_\theta^N + B_\theta^S$)

$$\begin{aligned} B_\theta^N &= (F_{11} + F_{31}r^2 + F_{51}r^4)\cos\theta \\ &+ (2F_{22}r + 2F_{42}r^3 + 2F_{62}r^5)\cos 2\theta \\ &+ (3F_{33}r^2 + 3F_{53}r^4)\cos 3\theta \\ &+ (4F_{44}r^3 + 4F_{64}r^5)\cos 4\theta \\ &+ 5F_{55}r^4 \cos 5\theta \\ &+ 6F_{66}r^5 \cos 6\theta \end{aligned} \tag{60}$$

$$
\begin{aligned}
B_\theta^S &= -(G_{11} + G_{31}r^2 + G_{51}r^4)\sin\theta \\
&- (2G_{22}r + 2G_{42}r^3 + 2G_{62}r^5)\sin 2\theta \\
&- (3G_{33}r^2 + 3G_{53}r^4)\sin 3\theta \\
&- (4G_{44}r^3 + 4G_{64}r^5)\sin 4\theta \\
&- 5G_{55}r^4 \sin 5\theta \\
&- 6G_{66}r^5 \sin 6\theta
\end{aligned} \tag{61}
$$

3.3.3 B_z expansion ($B_z = B_z^N + B_z^S$)

$$
\begin{aligned}
B_z^N &= (F'_{11}r + F'_{31}r^3 + F'_{51}r^5)\sin\theta \\
&+ (F'_{22}r^2 + F'_{42}r^4)\sin 2\theta \\
&+ (F'_{33}r^3 + F'_{53}r^5)\sin 3\theta \\
&+ F'_{44}r^4 \sin 4\theta \\
&+ F'_{55}r^5 \sin 5\theta
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
B_z^S &= G'_{00} + G'_{20}r^2 + G'_{40}r^4 \\
&+ (G'_{11}r + G'_{31}r^3 + G'_{51}r^5)\cos\theta \\
&+ (G'_{22}r^2 + G'_{42}r^4)\cos 2\theta \\
&+ (G'_{33}r^3 + G'_{53}r^5)\cos 3\theta \\
&+ G'_{44}r^4 \cos 4\theta \\
&+ G'_{55}r^5 \cos 5\theta
\end{aligned} \tag{63}
$$

4 Comparison of Coefficients

Let us now compare the coefficients in the rectangular and cylindrical coordinate expansions. To do this we first note that

$$r^l \cos m\theta = \frac{r^{l-m}}{2}\{(x+iy)^m + (x-iy)^m\}, \tag{64}$$

$$r^l \sin m\theta = \frac{r^{l-m}}{2i}\{(x+iy)^m - (x-iy)^m\}. \tag{65}$$

Thus we have

$$r\cos\theta = x, \quad r\sin\theta = y \tag{66}$$

$$r^2 = x^2 + y^2, \quad r^2 \cos 2\theta = x^2 - y^2, \quad r^2 \sin 2\theta = 2xy \tag{67}$$

$$r^3 \cos\theta = x^3 + xy^2, \quad r^3 \cos 3\theta = x^3 - 3xy^2 \tag{68}$$

$$r^3 \sin\theta = x^2 y + y^3, \quad r^3 \sin 3\theta = 3x^2 y - y^3 \tag{69}$$

$$r^4 = x^4 + 2x^2y^2 + y^4, \quad r^4 \cos 2\theta = x^4 - y^4, \quad r^4 \cos 4\theta = x^4 - 6x^2y^2 + y^4 \tag{70}$$

$$r^4 \sin 2\theta = 2x^3 y + 2xy^3, \quad r^4 \sin 4\theta = 4x^3 y - 4xy^3 \tag{71}$$

$$r^5 \cos\theta = x^5 + 2x^3y^2 + xy^4, \quad r^5 \cos 3\theta = x^5 - 2x^3y^2 - 3xy^4 \tag{72}$$

$$r^5 \cos 5\theta = x^5 - 10x^3y^2 + 5xy^4 \tag{73}$$

$$r^5 \sin\theta = x^4 y + 2x^2y^3 + y^5, \quad r^5 \sin 3\theta = 3x^4 y + 2x^2y^3 - y^5 \tag{74}$$

$$r^5 \sin 5\theta = 5x^4 y - 10x^2y^3 + y^5 \tag{75}$$

$$r^6 = x^6 + 3x^4y^2 + 3x^2y^4 + y^6, \quad r^6 \cos 6\theta = x^6 - 15x^4y^2 + 15x^2y^4 - y^6 \tag{76}$$

$$r^6 \cos 2\theta = x^6 + x^4y^2 - x^2y^4 - y^6, \quad r^6 \cos 4\theta = x^6 - 5x^4y^2 - 5x^2y^4 + y^6 \tag{77}$$

$$r^6 \sin 6\theta = 6x^5 y - 20x^3y^3 + 6xy^5 \tag{78}$$

$$r^6 \sin 2\theta = 2x^5 y + 4x^3y^3 + 2xy^5, \quad r^6 \sin 4\theta = 4(x^5 y - xy^5). \tag{79}$$

Using these expressions in (31) and equating terms order-by-order in the rectangular and cylindrical expansions of the potential, we find

$$C_{00} = G_{00}, \quad C_{01}x + C_{10}y = G_{11}x + F_{11}y \tag{80}$$

$$C_{11}xy = F_{22}(2xy) \tag{81}$$

$$\frac{1}{2}C_{02}x^2 + \frac{1}{2}C_{20}y^2 = G_{20}(x^2 + y^2) + G_{22}(x^2 - y^2) \tag{82}$$

$$\frac{1}{2}C_{12}x^2 y + \frac{1}{6}C_{30}y^3 = F_{31}(x^2 y + y^3) + F_{33}(3x^2 y - y^3) \tag{83}$$

$$\frac{1}{2}C_{21}xy^2 + \frac{1}{6}C_{03}x^3 = G_{31}(x^3 + xy^2) + G_{33}(x^3 - 3xy^2) \tag{84}$$

$$\frac{1}{6}(C_{13}x^3y + C_{31}xy^3) = F_{42}(2x^3y + 2xy^3) + F_{44}(4x^3y - 4xy^3) \tag{85}$$

$$\begin{aligned} \frac{1}{24}(C_{04}x^4 + 6C_{22}x^2y^2 + C_{40}y^4) &= G_{40}(x^4 + 2x^2y^2 + y^4) \\ &+ G_{42}(x^4 - y^4) \\ &+ G_{44}(x^4 - 6x^2y^2 + y^4) \end{aligned} \tag{86}$$

$$\begin{aligned} \frac{1}{120}(5C_{14}x^4y + 10C_{32}x^2y^3 + C_{50}y^5) &= F_{51}(x^4y + 2x^2y^3 + y^5) \\ &+ F_{53}(3x^4y + 2x^2y^3 - y^5) \\ &+ F_{55}(5x^4y - 10x^2y^3 + y^5) \end{aligned} \tag{87}$$

$$\begin{aligned} \frac{1}{120}(C_{05}x^5 + 10C_{23}x^3y^2 + 5C_{41}xy^4) &= G_{51}(x^5 + 2x^3y^2 + xy^4) \\ &+ G_{53}(x^5 - 2x^3y^2 - 3xy^4) \\ &+ G_{55}(x^5 - 10x^3y^2 + 5xy^4) \end{aligned} \tag{88}$$

$$\begin{aligned} \frac{1}{720}(6C_{15}x^5y + 20C_{33}x^3y^3 + 6C_{51}xy^5) &= \\ F_{62}(2x^5y + 4x^3y^3 + 2xy^5) &+ \\ F_{64}(4x^5y - 4xy^5) &+ \\ F_{66}(6x^5y - 20x^3y^3 + 6xy^5). \end{aligned} \tag{89}$$

Thus we have

$$C_{00} = G_{00}, \quad B_0 = C_{10} = F_{11}, \quad A_0 = C_{01} = G_{11} \tag{90}$$

$$B_1 = C_{11} = 2F_{22}, \quad A_1 = C_{02} = 2(G_{20} + G_{22}), \quad C_{20} = 2(G_{20} - G_{22}) \tag{91}$$

$$B_2 = C_{12} = 2(F_{31} + 3F_{33}), \quad C_{30} = 6(F_{31} - F_{33}) \tag{92}$$

$$A_2 = C_{03} = 6(G_{31} + G_{33}), \quad C_{21} = 2(G_{31} - 3G_{33}) \tag{93}$$

$$B_3 = C_{13} = 6(2F_{42} + 4F_{44}), \quad C_{31} = 6(2F_{42} - 4F_{44}) \tag{94}$$

$$A_3 = C_{04} = 24(G_{40} + G_{42} + G_{44}), \quad C_{40} = 24(G_{40} - G_{42} + G_{44}) \tag{95}$$

$$C_{22} = 4(2G_{40} - 6G_{44}) \tag{96}$$

$$B_4 = C_{14} = 24(F_{51} + 3F_{53} + 5F_{55}), \quad C_{50} = 120(F_{51} - F_{53} + F_{55}) \tag{97}$$

$$C_{32} = 12(2F_{51} + 2F_{53} - 10F_{55}) \tag{98}$$

$$C_{41} = 24(G_{51} - 3G_{53} + 5G_{55}), \quad A_4 = C_{05} = 120(G_{51} + G_{53} + G_{55}) \tag{99}$$

$$C_{23} = 12(2G_{51} - 2G_{53} - 10G_{55}) \tag{100}$$

$$B_5 = C_{15} = 120(2F_{62} + 4F_{64} + 6F_{66}), \quad C_{51} = 120(2F_{62} - 4F_{64} + 6F_{66}) \tag{101}$$

$$C_{33} = 36(4F_{62} - 20F_{66}). \tag{102}$$

These equations show that the multipole coefficients, $B_n = C_{1n}$ and $A_n = C_{0,n+1}$, generally contain, in addition to the coefficients $F_{n+1,n+1}$ and $G_{n+1,n+1}$ respectively, coefficients $F_{n+1,m}$ and $G_{n+1,m}$ with $m < n+1$. Since the coefficients with $m < n+1$ may all be expressed in terms of derivatives (with respect to z) of the coefficients F_{mm} and G_{mm}, we see that when the fields are independent of z, we have

$$B_n = C_{1n} = (n+1)!\, F_{n+1,n+1} \tag{103}$$

and

$$A_n = C_{0,n+1} = (n+1)!\, G_{n+1,n+1}. \tag{104}$$

Thus, since the $F_{n+1,n+1}$ and $G_{n+1,n+1}$ coefficients arise from magnets with $2(n+1)$ poles, the coefficients B_n and A_n are called $2(n+1)$-pole coefficients.

Equations (90–102) must, of course, be consistent with the recursion relation (6). As a check of our algebra, we note that

$$C_{20} + C_{02} = 4G_{20} = -G''_{00} = -C''_{00} \tag{105}$$

$$C_{12} + C_{30} = 8F_{31} = -F''_{11} = -C''_{10} \tag{106}$$

$$C_{21} + C_{03} = 8G_{31} = -G''_{11} = -C''_{01} \tag{107}$$

$$C_{13} + C_{31} = 24F_{42} = -2F''_{22} = -C''_{11} \tag{108}$$

$$C_{40} + C_{22} = 32G_{40} - 24G_{42} = -2G''_{20} + 2G''_{22} = -C''_{20} \tag{109}$$

$$C_{04} + C_{22} = 32G_{40} + 24G_{42} = -2G''_{20} - 2G''_{22} = -C''_{02} \tag{110}$$

$$C_{14} + C_{32} = 48F_{51} + 96F_{53} = -2F''_{31} - 6F''_{33} = -C''_{12} \tag{111}$$

$$C_{50} + C_{32} = 144F_{51} - 96F_{53} = -6F''_{31} + 6F''_{33} = -C''_{30} \tag{112}$$

$$C_{41} + C_{23} = 48G_{51} - 96G_{53} = -2G''_{31} + 6G''_{33} = -C''_{21} \tag{113}$$

$$C_{05} + C_{23} = 144G_{51} + 96G_{53} = -6G''_{31} - 6G''_{33} = -C''_{03} \tag{114}$$

$$C_{51} + C_{33} = 384F_{62} - 480F_{64} = -12F''_{42} + 24F''_{44} = -C''_{31} \tag{115}$$

$$C_{15} + C_{33} = 384F_{62} + 480F_{64} = -12F''_{42} - 24F''_{44} = -C''_{13}. \tag{116}$$

5 Magnet with Dipole Symmetry

We define a magnet with dipole symmetry to have a magnetic potential such that

$$\phi(x, -y, z) = -\phi(x, y, z), \quad \phi(r, -\theta, z) = -\phi(r, \theta, z) \tag{117}$$

and

$$\phi(-x, y, z) = \phi(x, y, z), \quad \phi(r, \pi - \theta, z) = \phi(r, \theta, z). \tag{118}$$

It then follows from (31) that the skew coefficients G_{lm} and all normal coefficients except F_{l1}, F_{l3}, F_{l5}, F_{l7}, …, etc. are zero. Thus we have

$$\phi(r, \theta, z) = \sum_{l=1}^{\infty} \sum_{k=0}^{\infty} \{F_{l,2k+1} \sin(2k+1)\theta\} r^l. \tag{119}$$

Equations (58–63) then become, to 5th order in r,

$$\begin{aligned} B_r &= (F_{11} + 3F_{31}r^2 + 5F_{51}r^4)\sin\theta \\ &+ (3F_{33}r^2 + 5F_{53}r^4)\sin 3\theta \\ &+ 5F_{55}r^4 \sin 5\theta \end{aligned} \tag{120}$$

$$\begin{aligned} B_\theta &= (F_{11} + F_{31}r^2 + F_{51}r^4)\cos\theta \\ &+ (3F_{33}r^2 + 3F_{53}r^4)\cos 3\theta \\ &+ 5F_{55}r^4 \cos 5\theta \end{aligned} \tag{121}$$

$$\begin{aligned} B_z &= (F'_{11}r + F'_{31}r^3 + F'_{51}r^5)\sin\theta \\ &+ (F'_{33}r^3 + F'_{53}r^5)\sin 3\theta \\ &+ F'_{55}r^5 \sin 5\theta \end{aligned} \tag{122}$$

where

$$F_{31} = -\frac{1}{8}F''_{11}, \quad F_{51} = -\frac{1}{24}F''_{31} = \frac{1}{192}F''''_{11}, \quad F_{53} = -\frac{1}{16}F''_{33}. \tag{123}$$

Putting the F_{lm} coefficients that are zero into (90–102) we find that

$$B_1 = B_3 = B_5 = 0. \tag{124}$$

The field expansion to 5th order in rectangular coordinates is then

$$\begin{aligned} B_x &= B_2 xy \\ &+ B_4(x^3y - xy^3)/6 - B''_2 xy^3/6 \end{aligned} \tag{125}$$

$$\begin{aligned} B_y &= B_0 \\ &+ B_2(x^2 - y^2)/2 - B''_0 y^2/2 \\ &+ B_4(x^4 - 6x^2y^2 + y^4)/24 \\ &+ B''_2(y^4 - 3x^2y^2)/12 + B''''_0 y^4/24 \end{aligned} \tag{126}$$

$$\begin{aligned} B_z &= B'_0 y \\ &+ B'_2(3x^2y - y^3)/6 - B'''_0 y^3/6 \\ &+ B'_4(5x^4y - 10x^2y^3 + y^5)/120 \\ &+ B'''_2(y^5 - 5x^2y^3)/60 + B'''''_0 y^5/120 \end{aligned} \tag{127}$$

where

$$B_0 = C_{10} = F_{11}, \tag{128}$$

$$B_2 = C_{12} = 2(F_{31} + 3F_{33}) = 2\left(3F_{33} - \frac{1}{8}F_{11}''\right), \tag{129}$$

$$B_4 = C_{14} = 24(F_{51} + 3F_{53} + 5F_{55}) = 24\left(\frac{1}{192}F_{11}'''' - \frac{3}{16}F_{33}'' + 5F_{55}\right). \tag{130}$$

6 Magnet with Quadrupole Symmetry

We define a magnet with quadrupole symmetry to have a magnetic potential such that

$$\phi(x,-y,z) = -\phi(x,y,z), \quad \phi(r,-\theta,z) = -\phi(r,\theta,z), \tag{131}$$

$$\phi(-x,y,z) = -\phi(x,y,z), \quad \phi(r,\pi-\theta,z) = -\phi(r,\theta,z), \tag{132}$$

and

$$\phi(y,x,z) = \phi(x,y,z), \quad \phi(r,\pi/2-\theta,z) = \phi(r,\theta,z). \tag{133}$$

It then follows from (31) that the skew coefficients, G_{lm}, and all normal coefficients except F_{l2}, F_{l6}, $F_{l,10}$, $F_{l,14}$, ..., etc. are zero. Thus we have

$$\phi(r,\theta,z) = \sum_{l=1}^{\infty}\sum_{k=0}^{\infty}\{F_{l,4k+2}\sin(4k+2)\theta\}r^l \tag{134}$$

and to 6th order in r

$$\begin{aligned} B_r &= (2F_{22}r + 4F_{42}r^3 + 6F_{62}r^5)\sin 2\theta \\ &+ 6F_{66}r^5 \sin 6\theta \end{aligned} \tag{135}$$

$$\begin{aligned} B_\theta &= (2F_{22}r + 2F_{42}r^3 + 2F_{62}r^5)\cos 2\theta \\ &+ 6F_{66}r^5 \cos 6\theta \end{aligned} \tag{136}$$

$$\begin{aligned} B_z &= (F_{22}'r^2 + F_{42}'r^4)\sin 2\theta \\ &+ F_{66}'r^6 \sin 6\theta \end{aligned} \tag{137}$$

where

$$F_{42} = -\frac{1}{12} F_{22}'', \quad F_{62} = -\frac{1}{32} F_{42}'' = \frac{1}{384} F_{22}''''. \tag{138}$$

Putting the zero and nonzero F_{lm} coefficients into (90–102) we find that

$$B_0 = B_2 = B_4 = 0 \tag{139}$$

and

$$B_1 = C_{11} = 2F_{22}, \quad B_3 = C_{13} = 12F_{42} = -F_{22}'', \tag{140}$$

$$B_5 = C_{15} = 120(2F_{62} + 6F_{66}) = 120\left(6F_{66} + \frac{1}{192} F_{22}''''\right). \tag{141}$$

The field expansion to 5th order in rectangular coordinates is then

$$\begin{aligned} B_x &= B_1 y \\ &+ B_3(3x^2y - y^3)/6 - B_1'' y^3/6 \\ &+ B_5(5x^4y - 10x^2y^3 + y^5)/120 \\ &+ B_3''(y^5 - 5x^2y^3)/60 + B_1'''' y^5/120 \end{aligned} \tag{142}$$

$$\begin{aligned} B_y &= B_1 x \\ &+ B_3(x^3 - 3xy^2)/6 - B_1'' xy^2/2 \\ &+ B_5(x^5 - 10x^3y^2 + 5xy^4)/120 \\ &+ B_3''(xy^4 - x^3y^2)/12 + B_1'''' xy^4/24 \end{aligned} \tag{143}$$

$$\begin{aligned} B_z &= B_1' xy \\ &+ B_3'(x^3y - xy^3)/6 - B_1''' xy^3/6 \end{aligned} \tag{144}$$

Note that

$$B_3 = -B_1''/2, \tag{145}$$

so this coefficient is due to the second derivative of the quadrupole coefficient.

7 Magnet with Solenoid Symmetry

For a magnet with pure solenoid symmetry, the potential is independent of θ and we therefore have

$$\phi(r, \theta, z) = \sum_{l=0}^{\infty} G_{2l,0}\, r^{2l}. \tag{146}$$

This shows the physical significance of the coefficients $G_{2l,0}$. The field components in this case are

$$B_\theta = 0, \quad B_r = 2G_{20}r + 4G_{40}r^3 + 6G_{60}r^5 + \cdots, \tag{147}$$

$$B_z = G'_{00} + G'_{20}r^2 + G'_{40}r^4 + \cdots \tag{148}$$

where

$$G_{20} = -\frac{1}{4}G''_{00}, \quad G_{40} = -\frac{1}{16}G''_{20}, \quad G_{60} = -\frac{1}{36}G''_{40}. \tag{149}$$

8 References

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