# *Electric fields of a uniformly charged elliptical beam*

George Parzen

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COLLIDER-ACCELERATOR DEPARTMENT
BROOKHAVEN NATIONAL LABORATORY
UPTON, NEW YORK  11973

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## Introduction

This paper presents results for the electric field due to a uniformly charged elliptical beam outside the beam. Results for the field inside inside the beam are well known [1, 2] The beam being considered extends indefinitly in the $z$ direction and has an elliptical boundary in $x$ and $y$ given by

$$x^2/a^2 + y^2/b^2 = 1 \tag{1}$$

The charge density, $\rho(x, y, z)$ is uniform within the elliptical boundary, zero outside the elliptical boundary, and does not depend on $z$. The results given below depend on the observation made by B. Houssais [3], that the result for the electric field of a gaussian charge distribution given by W. Kellog [1] as a one dimensional integral would hold for any elliptical charge distribution as defined below. This may be stated as follows. Let the charge distribution be given as

$$\rho(x, y, z) = \lambda n(x, y) \tag{2}$$

where $\lambda$ is the charge per unit length and

$$\int dx dy \, n(x, y) = 1 \tag{3}$$

A charge distribution will be called elliptical if $n(x, y)$ can be written as

$$
\begin{aligned}
n(x, y) &= \hat{n}(T)/\pi ab \\
T &= x^2/a^2 + y^2/b^2
\end{aligned} \tag{4}
$$

1

For the uniform elliptical beam , $\hat{n}(T)$ is given by

$$\begin{align}
\hat{n}(T) &= 1, \; T \le 1 \\
\hat{n}(T) &= 0, \; T > 1
\end{align} \tag{5}$$

For a Gaussian beam, $\hat{n}(T)$ is given by

$$\hat{n}(T) = exp(-T) \tag{6}$$

One can show, using Eq. 3, that $\hat{n}(T)$ obeys the equation

$$\int_0^\infty dT \; \hat{n}(T) = 1 \tag{7}$$

The generalization of the Kellog result for any elliptical beam is then

$$\begin{align}
E_x &= 2\lambda \int_0^\infty dt \frac{\hat{n}(\hat{T})}{(a^2 + t)^{3/2}(b^2 + t)^{1/2}} \\
\hat{T} &= x^2/(a^2 + t) + y^2/(b^2 + t)
\end{align} \tag{8}$$

A similar result, with $a, b$ and $x, y$ interchanged will give $E_y$

## Electric fields for x,y inside the beam

As a first step, the fields inside a uniformly charged elliptical beam will be found using Eq. 8 .In this case, $\hat{T}$ is always $\le 1$ since for $t = 0$ , $\hat{T} = x^2/a^2 + y^2/b^2$, which is $\le 1$ for x,y inside the beam, and decreases further for larger t. Eq. 8 then becomes

$$\begin{align}
E_x &= 2\lambda x \int_0^\infty dt \frac{1}{(a^2 + t)^{3/2}(b^2 + t)^{1/2}} \\
\hat{T} &= x^2/(a^2 + t) + y^2/(b^2 + t)
\end{align} \tag{9}$$

The integral in Eq. 9 can be done using the result

$$\int_{t_1}^\infty dt \frac{1}{(a^2 + t)^{3/2}(b^2 + t)^{1/2}} = 2\frac{1}{(a^2 + t_1)^{1/2}} \frac{1}{(a^2 + t_1)^{1/2} + (b^2 + t_1)^{1/2}} \tag{10}$$

This gives

$$E_x = 4\lambda x \frac{1}{a(a + b)} \tag{11}$$

and a similar result for $E_y$ with a and b interchanged and x replaced by $y$.

2

# Electric fields outside the beam when $y = 0$

As the next step, the fields outside a uniformly charged elliptical beam will be found using Eq. 8 for the case when $y = 0$. The results in this case are simpler and the mathematics is easier to comprehend. In this case, $\hat{T}$ is $> 1$ for $t = 0$ since for $t = 0$ , $\hat{T} = x^2/a^2 + y^2/b^2$, which is $> 1$ for x,y outside the beam. For larger $t$, $\hat{T}$ decreases and reaches the vaue of 1 at $t = t_1$, and at still larger $t$, $\hat{T}$ decreases further always remaining smaller than 1.The integral in Eq. 8 then goes from $t = t_1$, to $t = \infty$. Eq. 8 then becomes

$$
\begin{aligned}
E_x &= 2\lambda x \int_{t_1}^{\infty} dt \frac{1}{(a^2 + t)^{3/2}(b^2 + t)^{1/2}} \\
\hat{T} &= x^2/(a^2 + t) \\
t_1 &= x^2 - a^2 \\
y &= 0
\end{aligned}
$$

(12)

(13)

Using Eq. 10. one finds

$$
\begin{aligned}
E_x &= 4\lambda \frac{1}{x + (x^2 + b^2 - a^2)^{1/2}} \\
E_y &= 0 \\
y &= 0
\end{aligned}
$$

(14)

$E_{xx} = \partial E_x/\partial x$ is given by

$$
E_{xx} = -\frac{E_x}{(x^2 + b^2 - a^2)^{1/2}}
$$

(15)

# Electric fields outside the beam when $y \neq 0$

As the final step, the fields outside a uniformly charged elliptical beam will be found using Eq. 8 for the general case. In this case, $\hat{T}$ is $> 1$ for $t = 0$ since for $t = 0$ , $\hat{T} = x^2/a^2 + y^2/b^2$, which is $> 1$ for x,y outside the beam. For larger $t$, $\hat{T}$ decreases and reaches the vaue of 1 at $t = t_1$, and at still larger $t$, $\hat{T}$ decreases further always remaining smaller than 1.The integral in Eq. 8 then goes from $t = t_1$, to $t = \infty$. Eq. 8 then becomes

$$
\begin{aligned}
E_x &= 2\lambda x \int_{t_1}^{\infty} dt \frac{1}{(a^2 + t)^{3/2}(b^2 + t)^{1/2}} \\
x^2/(a^2 + t_1) + y^2/(b^2 + t_1) &= 1
\end{aligned}
$$

(16)

$t_1$ is the positive root of the equation

$$x^2/(a^2 + t_1) + y^2/(b^2 + t_1) \;=\; 1 \tag{17}$$

The quadratic equation for $t_1$, Eq. 17, can be solved to give

$$
\begin{aligned}
t_1 &= (B^2/4 + C)^{1/2} + B/2 \\
B &= x^2 + y^2 - a^2 - b^2 \\
C &= x^2 b^2 + y^2 a^2 - a^2 b^2
\end{aligned}
\tag{18}
$$

Eq. 16 gives the result for $E_x$

$$E_x \;=\; 4\lambda x \frac{1}{(a^2 + t_1)^{1/2}} \frac{1}{(a^2 + t_1)^{1/2} + (b^2 + t_1)^{1/2}} \tag{19}$$

and a similar result for $E_y$ with a and b interchanged and $x$ replaced by $y$.

It may be usefull to also have results for the derivatives of the fields, $E_{xx}, E_{yy}, E_{xy} = E_{yx}$, where $E_{xx} = \partial E_x/\partial x$, $E_{yy} = \partial E_y/\partial y$ and $E_{xy} = \partial E_x/\partial y$. $E_{xx}$ is found using Eq. 16 for $E_x$

$$E_{xx} \;=\; \frac{E_x}{x} - 2\lambda x \frac{1}{(a^2 + t_1)^{3/2}(b^2 + t_1)^{1/2}} \frac{dt_1}{dx} \tag{20}$$

$dt_1/dx$ can be found from Eq. 17 for $t_1$ as

$$\frac{dt_1}{dx} \;=\; 2x \frac{(a^2 + t_1)(b^2 + t_1)^2}{x^2(b^2 + t_1)^2 + y^2(a^2 + t_1)^2} \tag{21}$$

This gives for $E_{xx}$

$$E_{xx} \;=\; \frac{E_x}{x} - 4\lambda x^2 \frac{(a^2 + t_1)^{-1/2}(b^2 + t_1)^{3/2}}{x^2(b^2 + t_1)^2 + y^2(a^2 + t_1)^2} \tag{22}$$

$E_{yy}$ and $dt_1/dy$ can be found by interchanging x and y, and a and b. $E_{xy}$ can be found in the same way as

$$E_{xy} \;=\; -4\lambda xy \frac{(a^2 + t_1)^{1/2}(b^2 + t_1)^{1/2}}{x^2(b^2 + t_1)^2 + y^2(a^2 + t_1)^2} \tag{23}$$

# References

[1] W.Kellog, Foundations of Potential Theory, (Dover Publications,New York,1953), p. 192.

[2] L. Teng, Report ANLAD-59 (1963)

[3] F.J. Sacherer, PAC71, p.1105, (1971)